Angles in Complex Vector Spaces

K. Scharnhorst †

Humboldt-Universität zu Berlin Institut für Physik Invalidenstr. 110 D-10115 Berlin Federal Republic of Germany

Abstract

The article reviews some of the (fairly scattered) information available in the mathematical literature on the subject of angles in complex vector spaces. The following angles and their relations are considered: Euclidean, complex, and Hermitian angles, (Kasner's) pseudo-angle, the Kähler angle (synonyms for the latter used in the literature are: angle of inclination, characteristic deviation, holomorphic deviation, holomorphy angle, Wirtinger angle, slant angle).

[†]E-mail: scharnh@physik.hu-berlin.de

1. Introduction. The angle between two vectors in a real vector space is a concept often already introduced to students at the school level. Complex vector spaces feature prominently in most linear algebra courses at the undergraduate level and they can be found in many branches of mathematics, the natural sciences, and engineering. Therefore, it is surprising that very little guidance is available in the mathematical literature on angles in complex vector spaces. Even advanced books on linear algebra or higher geometry hardly mention the subject. In order to find some information on it (which is widely scattered, however) one has to resort almost exclusively to the journal literature. No review seems to be available, closest to this come only sections in the recently published monographs by Rosenfeld ([1], chap. III, §3.3, sect. 3.3.6, pp. 182/183) and, more detailed, by Goldman ([2], chap. 2, sect. 2.2.2, pp. 36-39) which, however, are supplemented by no or only very few references, respectively, concerning the subject (the latter monograph appeared in print only when the first version of the present paper had been put into circulation on the Los Alamos Math Archive). In the present article, which grew from the working needs of a theoretical physicist, we undertake to fill this gap for a rather general audience.

To begin with, let us consider the problem one faces in introducing the concept of an angle in complex vector spaces. In any (finite-dimensional) real (Euclidean) vector space $V_{\mathbf{R}}$ ($\simeq \mathbf{R}_m$, $m \in \mathbf{N}$, $m \geq 2$) equipped with the scalar product $(A, B)_{\mathbf{R}} = \sum_{k=1}^m A_k B_k$ for any pair of vectors $A, B \in V_{\mathbf{R}}$ one can define an (real) angle $\Theta(A, B)$, $0 \leq \Theta \leq \pi$, between these two vectors by means of the standard formula ($|A| = \sqrt{(A, A)_{\mathbf{R}}}$)

$$\cos\Theta(A,B) = \frac{(A,B)_{\mathbf{R}}}{|A||B|} . \tag{1}$$

The introduction of an angle between two vectors a, b of a (finite-dimensional) complex (Hermitian, unitary) vector space $V_{\mathbf{C}}$ ($\simeq \mathbf{C}_n, n \in \mathbf{N}, n \geq 2$) is ambiguous and can be performed

- I. either directly in the complex vector space $V_{\mathbf{C}}$ by relying on the Hermitian product $(a,b)_{\mathbf{C}} = \sum_{k=1}^{n} \bar{a}_k b_k$ defined in it for any pair of vectors $a,b \in V_{\mathbf{C}}$ (\bar{a}_k denotes the complex conjugate of $a_k \in \mathbf{C}$), or
- II. by relying on the real vector space $V_{\mathbf{R}} \ (\simeq \mathbf{R}_{2n})$ isometric to $V_{\mathbf{C}}$.

Both approaches which are not completely independent are equally justified. Consequently, one has to study their relation and this is what this article mainly is concerned with.

2. The Euclidean, complex, Hermitian, and pseudo-angles. The (real-valued) Euclidean angle between two vectors $a, b \in V_{\mathbf{C}}$ related to item II. and denoted by $\Theta(a, b), 0 \leq \Theta \leq \pi$, is defined by the formula

$$\cos\Theta(a,b) = \cos\Theta(A,B) = \frac{(A,B)_{\mathbf{R}}}{|A||B|}, \qquad (2)$$

where we choose to determine the components of the vectors A, B in the real vector space $V_{\mathbf{R}}$ by means of the relations $A_{2k-1} = \operatorname{Re} a_k$ and $A_{2k} = \operatorname{Im} a_k$, $k = 1, \ldots, n$. On the other hand, related to item I., a complex (-valued) angle $\Theta_{\mathbf{c}}(a, b)$ can be introduced by means of the relation ($|a| = \sqrt{(a, a)_{\mathbf{C}}} = |A|$)

$$\cos \Theta_{c}(a,b) = \frac{(a,b)_{\mathbf{C}}}{|a||b|} . \tag{3}$$

Here, we follow the definition applied in the majority of the literature on the subject (cf., e. g., [3], chap. VI, §2, eq. (6.65), p. 574; incidentally, note that also other views can be found, cf. [4]). Furthermore, one can write eq. (3) as

$$\cos \Theta_{c}(a,b) = \rho e^{i\varphi} , \qquad (4)$$

where $(\rho \le 1)^2$

$$\rho = \cos \Theta_{H}(a, b) = |\cos \Theta_{c}(a, b)| . \tag{5}$$

 $\Theta_{\rm H}(a,b),\,0\leq\Theta_{\rm H}\leq\frac{\pi}{2}$, is called the Hermitian angle³ between the vectors $a,b\in V_{\bf C}$ while $\varphi=\varphi(a,b),\,-\pi\leq\varphi\leq\pi$ is called their (Kasner's) pseudo-angle [13]-[16]⁴. The pseudo-angle φ is of relevance in the context of pseudoconformal transformations. However, if one is interested just in the angle between two lines in the complex vector space $V_{\bf C}$ which are given by the vectors $a,b\in V_{\bf C}$ the concept of the pseudo-angle can be disregarded. As any line in $V_{\bf C}$ defined by the vector a can also be be given in terms of any other vector a'=za ($z\in{\bf C},\,z\neq0$) the pseudo-angle φ does not have any meaning in this context. In other words, it can be disregarded in

¹Throughout the article we denote vectors in the real vector space $V_{\mathbf{R}}$ by capital letters and vectors in the complex vector space $V_{\mathbf{C}}$ by small letters.

²This follows from the Cauchy inequality $(a,b)_{\mathbf{C}}(b,a)_{\mathbf{C}} \leq (a,a)_{\mathbf{C}}(b,b)_{\mathbf{C}}$ which can be derived from $0 \leq (a - \sigma b, a - \sigma b)_{\mathbf{C}}$, $\sigma = (b,a)_{\mathbf{C}}/|b|^2$ ([3], chap. VI, §2, eq. (6.67), p. 574).

³This term (not the concept itself, however) has apparently been used first in [5], §I., sect. 4, p. 96, also see [6], [7] (historically, there have been minor differences in the definition used by some Italian authors, see, e.g., footnote 3 in [6], p. 395). For some early work on angles in complex vector spaces see [8]-[12] (for further references see the reprint volume cited in [12]).

⁴In a somewhat different context, the term *pseudo-angle* has already earlier been used by Giraud [17], pp. 68/69, who calls this quantity the *second pseudo-angle* while he denotes the Hermitian angle by the term *first pseudo-angle* (see [2], chap. 2, sect. 2.2.2, p. 38).

complex projective spaces \mathbb{CP}^n . The Fubini-Study metric employed in such spaces is just given by the cosine of the Hermitian angle (5). The Hermitian angle can be understood geometrically as follows ([18], chap. IV, §32, pp. 405/406). As in any real vector space the cosine of the (Hermitian) angle between two vectors $a, b \in V_{\mathbb{C}}$ can be defined to be the ratio between the length of the orthogonal projection (with respect to the Hermitian product) of, say, the vector a onto the vector b to the length of the vector a itself (this projection vector is equal to a0 where a1 itself (this projection vector is equal to a2 where a3 itself (this projection vector is equal to a3 where a4 itself (this projection vector is equal to a4 where a5 itself (this projection vector is equal to a5 where a5 itself (this projection vector is equal to a6 where a5 itself (this projection vector is equal to a6 where a5 itself (this projection vector is equal to a6 where a5 itself (this projection vector is equal to a6 where a6 itself (this projection vector is equal to a6 where a6 itself (this projection vector is equal to a6 where a6 itself (this projection vector is equal to a6 where a6 itself (this projection vector is equal to a6 where a6 itself (this projection vector is equal to a6 where a6 itself (this projection vector is equal to a6 itself (th

3. The Kähler angle. In order to proceed further let us introduce the almost complex structure J, $J^2 = -1$, which acts as an operator in the real vector space $V_{\mathbf{R}}$ isometric to $V_{\mathbf{C}}$. In our coordinates the almost complex structure J performs the following transformations: $A_{2k-1} \longrightarrow A_{2k}$, $A_{2k} \longrightarrow -A_{2k-1}$, $k = 1, \ldots, n$. This is equivalent to the transformation $a \longrightarrow ia$ in $V_{\mathbf{C}}$. A subspace \mathcal{P} of $V_{\mathbf{R}}$ is called holomorphic, if it holds $\mathcal{P} = J\mathcal{P}$. It is called antiholomorphic⁶ (totally real, with a real Hermitian product), if it holds $\mathcal{P} \perp J\mathcal{P}$. Following the convention applied in a large fraction of the literature we introduce the notation $\tilde{A} = JA$, $A \in V_{\mathbf{R}}$ (note, $(\tilde{A}, B)_{\mathbf{R}} = -(A, \tilde{B})_{\mathbf{R}}$ for any two vectors $A, B \in V_{\mathbf{R}}$). By writing

$$\cos \Theta_{K}(a,b) \sin \Theta(a,b) = \cos \Theta_{K}(A,B) \sin \Theta(A,B) = \frac{(\tilde{A},B)_{\mathbf{R}}}{|A||B|}$$
 (6)

one can now introduce a further angle $\Theta_{\rm K}(a,b) = \Theta_{\rm K}(A,B)$, $0 \leq \Theta_{\rm K} \leq \pi$, which is called the Kähler angle⁷ between the vectors $a, b \in V_{\mathbb{C}}$, or the vectors $A, B \in V_{\mathbb{R}}$, respectively⁸. It is an intrinsic property of the (oriented) 2-plane in $V_{\mathbb{R}}$ defined by the (non-parallel) vectors A, B with respect to the almost complex structure J and, therefore, does not depend on the choice of the vectors in this plane ([23], sect. 3, p. 377, [26], [24], sect. 1, p. 65 (p. 83 of the English translation), [1], chap. III, §3.3, sect. 3.3.6, p. 182)⁹. The Kähler angle measures the deviation of a 2-plane from

⁵Note, that there is a misprint at the given location in [18], p. 406, 9th line from above: 'perpendikulyara' has to be replaced by the term 'vektora x'.

⁶Sometimes, this term is also used in a different sense.

⁷In using this term we follow the majority of the more recent mathematical literature (mostly, follow-up articles to [19]). The fundamental 2-form $\Phi(A,B)=(\tilde{A},B)_{\mathbf{R}}$ ([20], chap. IX, §4, p. 147, also see [21], §4, p. 182, eq. (4.15) and [20], chap. IX, §7, p. 167 for the definition of the corresponding (Kähler) angle) sometimes is referred to as the Kähler function (Kähler form). Other terms which are or have been used for the Kähler angle are: angle of inclination [22], §3, p. 369, [23], sect. 3, p. 378, [24], [25], characteristic deviation [26], [5], §I., sect. 4, p. 96, holomorphic deviation [27], holomorphy angle [1], chap. III, §3.3, sect. 3.3.6, pp. 182/183, [2], chap. 2, sect. 2.2.2, p. 36, Wirtinger angle, slant angle [28]-[30].

⁸If one wants to disregard the orientation of the 2-planes in $V_{\mathbf{R}}$ one can take on the r.h.s. of eq. (6) the absolute value restricting the Kähler angle to the interval $[0, \frac{\pi}{2}]$.

⁹For a 2-plane in $V_{\mathbf{R}}$ given by two unit vectors a, b orthogonal in the Euclidean sense $(\Theta(a, b) =$

holomorphicity. A holomorphic 2-plane has $\Theta_{\rm K}=0$, or $\Theta_{\rm K}=\pi$ (to see this consider a holomorphic system of vectors (J-basis) A, \tilde{A} of this 2-plane, $\tilde{A}=-A)^{10}$, while an antiholomorphic 2-plane has $\Theta_{\rm K}=\frac{\pi}{2}$. A 2-plane exhibiting some arbitrary Kähler angle $\Theta_{\rm K}$ is called $\Theta_{\rm K}$ -holomorphic [31] or general slant [28], [29]. If one is just interested in the Kähler angle of a 2-plane spanned by two vectors $A, B \in V_{\bf R}$ its defining equation (6) can be given a particularly simple shape if one chooses the vectors A, B such a way that they are orthogonal in the Euclidean sense $(\Theta(A, B) = \frac{\pi}{2})$.

4. Relations between the different angles. Now, applying unit vectors a, b $(|c| = |C| = |\tilde{C}| = 1, c = a, b)$ we can write eq. (3) as

$$\cos \Theta_{c}(a, b) = (A, B)_{\mathbf{R}} + i (\tilde{A}, B)_{\mathbf{R}}$$
$$= \cos \Theta(a, b) + i \cos \Theta_{K}(a, b) \sin \Theta(a, b) . \tag{7}$$

For the Hermitian angle Θ_H defined by eq. (5) one immediately finds from eq. (7) the following relation to the Euclidean angle Θ and the Kähler angle Θ_K ([5], sect. 4, p. 97, eq. (15)).

$$\sin \Theta_{\mathcal{H}}(a, b) = \sin \Theta_{\mathcal{K}}(a, b) \sin \Theta(a, b) \tag{8}$$

The pseudo-angle $\varphi(a, b)$ can also be linked to the other angles defined above. Using eqs. (4), (7), one obtains the following relations (also cf. [2], chap. 2, sect. 2.2.2, pp. $36-38)^{11}$.

$$\cos\Theta(a,b) = \cos\Theta_{H}(a,b) \cos\varphi(a,b) \tag{9}$$

$$\sin \varphi(a, b) = \cot \Theta_{K}(a, b) \tan \Theta_{H}(a, b) \tag{10}$$

$$\tan \varphi(a, b) = \cos \Theta_{K}(a, b) \tan \Theta(a, b)$$
 (11)

Specifically, one finds for any two vectors a, b of a complex line defining one and the same holomorphic 2-plane $(\Theta_{K}(a,b)=0)$ that the Hermitian angle vanishes $(\Theta_{H}(a,b)=0)$, this follows from eq. (8) and that the pseudo-angle is equal to the Euclidean angle $(\varphi(a,b)=\Theta(a,b))$, this follows from eq. (11); for $\Theta_{K}(a,b)=\pi$ holds

 $[\]frac{\pi}{2}$, $(a,b)_{\mathbf{C}} = \overline{-(b,a)_{\mathbf{C}}}$) one can convince oneself that $(a,b)_{\mathbf{C}}$ is invariant under the transformation $a' = \cos \phi \ a + \sin \phi \ b$, $b' = -\sin \phi \ a + \cos \phi \ b$, $\phi \in \mathbf{R}$.

¹⁰For a holomorphic 2-plane the action of the almost complex structure J consist in a rotation within this 2-plane by the (Euclidean) angle $\pi/2$ (= pseudo-angle, cf. sect. 4). In general, the almost complex structure J maps an arbitrary 2-plane in $V_{\mathbf{R}}$ to a 2-plane isoclinic to it (for an explanation of this term see sect. 5) with an angle which is equal to the Kähler angle of the original 2-plane ([24], sect. 1, p. 65 (p. 83 of the English translation), [2], chap. 1, sect. 1.3.3, p. 17).

¹¹Eqs. (10), (11) have been written in the most compact way ignoring the special cases to be treated separately when the tangent or the cotangent functions become infinite.

 $\varphi(a,b) = -\Theta(a,b)$). If the Hermitian angle $\Theta_{\rm H}(a,b)$ of two vectors a,b is different from $\pi/2$ and if, in addition, these vectors are orthogonal in the Euclidean sense $(\Theta(a,b)=\frac{\pi}{2})$ their pseudo-angle $\varphi(a,b)$ is either equal to $\pi/2$ or $-\pi/2$ (cf. eqs. (4), (7)). In the former case $\Theta_{\rm H}(a,b) = \Theta_{\rm K}(a,b)$ applies while in the latter case $\Theta_{\rm H}(a,b) = \pi - \Theta_{\rm K}(a,b)$ (cf. also [24], sect. 1, p. 65 (p. 83 of the English translation)). For any two vectors a,b defining an antiholomorphic 2-plane $(\Theta_{\rm K}(a,b)=\frac{\pi}{2})$ from eq. (7) one recognizes that the complex and the Euclidean angles coincide $(\Theta_{\rm c}(a,b)=\Theta(a,b))$ while for the Hermitian angle holds $\Theta_{\rm H}(a,b)=\Theta(a,b)$, or $\Theta_{\rm H}(a,b)=\pi-\Theta(a,b)$. The pseudo-angle $\varphi(a,b)$ between vectors of an antiholomorphic 2-plane is equal to 0, or $\pm \pi$, respectively.

5. Further comments. Before we continue the discussion with some further observations we need to introduce the concept of isoclinic 2-planes. Two 2-planes \mathcal{A} , \mathcal{B} in the real vector space $V_{\mathbf{R}}$ ($\simeq \mathbf{R}_{2n}$, $n \geq 2$) can intersect in various ways. In order to study their relation, to each pair of lines $\mathcal{X} \subset \mathcal{A}$, $\mathcal{Y} \subset \mathcal{B}$ the (Euclidean) angle they enclose can be calculated. Once a line $\mathcal{X} \subset \mathcal{A}$ is fixed, for any arbitrary line $\mathcal{Y} \subset \mathcal{B}$ the angle enclosed assumes values between some $\alpha_0 \geq 0$ ($\alpha_0 \leq \frac{\pi}{2}$) and $\pi/2$. In general, α_0 may lie between some minimal and some maximal value – the so-called stationary angles (principal angles) α_{\min} , α_{\max} – which are characteristic for the geometry of the pair of 2-planes \mathcal{A} , \mathcal{B} . If $\alpha_{\min} = \alpha_{\max}$, the 2-planes \mathcal{A} and \mathcal{B} are said to be (mutually) isoclinic and the isocliny angle $\alpha_0 = \alpha_{\min} = \alpha_{\max}$ of the two 2-planes can be determined by, say, calculating the (Euclidean) angle between some vector $X \in \mathcal{A}$ and its orthogonal projection onto \mathcal{B} (for a comprehensive list of references on the subject of stationary angles and isoclinic subspaces see sect. II of [32]).

After these definitions we can now proceed with our discussion. Any two holomorphic 2-planes in $V_{\mathbf{R}}$ (corresponding to two complex lines in $V_{\mathbf{C}}$) are isoclinic to each other ([33], §4, IX, p. 25, [34], §3, p. 34, [35], sect. 1-7, p. 51, theorem 1-7.4)¹². One can convince oneself that a vector of a holomorphic 2-plane in $V_{\mathbf{R}}$ and its (non-vanishing) orthogonal projection onto another (nonorthogonal) holomorphic 2-plane span an antiholomorphic 2-plane. Therefore, the isocliny angle α_0 between these two holomorphic 2-planes does not depend on the angle concept applied for its definition as the complex, the Euclidean, and the Hermitian angles agree for two lines of an antiholomorphic 2-plane. As mentioned in sect. 4, the Hermitian angle $\Theta_{\rm H}$ between two complex lines given by two vectors a, b which are orthogonal in the Euclidean

¹²This can easily be established by using holomorphic systems of unit vectors for each 2-plane given by a and b (A, \tilde{A} and B, \tilde{B} , respectively) which are chosen such that $(A,B)_{\mathbf{R}}=0$ holds. This way one can calculate with little effort the orthogonal projection of a given unit vector of one 2-plane onto the other one and find that the (Euclidean) angle between the vector and its projection is independent of the choice of the vector (the length (norm) of the projected vector is given by $|(\tilde{A},B)_{\mathbf{R}}|$).

sense is determined by their Kähler angle through the equations $\Theta_{\rm H}(a,b) = \Theta_{\rm K}(a,b)$ or $\Theta_{\rm H}(a,b) = \pi - \Theta_{\rm K}(a,b)$. Therefore, if the two holomorphic 2-planes are given in terms of vectors a,b which are orthogonal in the Euclidean sense the isocliny angle between these 2-planes can be calculated by evaluating the Kähler angle of these vectors.

Finally, we would like to add some comments on the literature concerning angles in vector spaces over fields. For complex vector spaces the relevant literature has been cited in course of the above discussion. The concept of the Kähler angle has also been extended to other, in particular quaternionic, vector spaces [23], [36]-[39], [6], [7], [24], [40], [41] (also see [25], sect. 6, p. 7 (p. 5 of the English translation)), [1], chap. III, §3.3, sect. 3.3.6, pp. 182/183. Moreover, the concept of the Kähler angle has also been generalized from 2-planes to higher dimensional linear subspaces of real vector spaces which stand in correspondence to complex and other vector spaces [27], [42]-[47]. For further considerations of angles in spaces over algebras see [3], chap. IV, p. 549, [48]-[51].

Acknowledgements. I am grateful to B. A. Rosenfeld for his comments on the first version of the paper and, in particular, for pointing me to a paper by Goldman which meanwhile has appeared in print as a book [2].

References

- [1] B. Rosenfeld, *Geometry of Lie Groups*. Mathematics and Its Applications, Vol. 393. Kluwer Academic Publishers, Dordrecht, 1997.
- [2] W. M. Goldman, *Complex Hyperbolic Geometry*. Oxford Mathematical Monographs. Clarendon Press, Oxford, 1999.
- [3] B. A. Rozenfel'd, Neevklidovy Geometrii [Non-Euclidean Geometries]. Gosudarstvennoe Isdatel'stvo Tekhniko-Teoreticheskoĭ Literatury, Moscow, 1955 [in Russian].
- [4] A. Froda, Sur l'angle complexe, orienté, de deux vecteurs d'un espace unitaire, Atti della Accademia Nazionale dei Lincei, Rendiconti, Classe di Scienze Fisiche, Matematiche e Naturali, 8. Ser., 30 (1961 (= Anno 358)) 845-853 [in French].
- [5] G. B. Rizza, Teoremi di curvatura in una V_{2n} quasi hermitiana, *Rivista di Matematica della Università di Parma*, 2. Ser., **2** (1961) 91-114 [in Italian].

- [6] M. Bruni, Misure angolari in uno spazio vettoriale quaternionale, *Rendiconti di Matematica e delle sue Applicazioni*, 5. Ser., **25** (1966) 394-401 [in Italian].
- [7] M. Bruni, Su alcune nozioni geometriche relative ad una varietà a struttura quaternionale generalizzata, Revue Roumaine de Mathématiques Pures et Appliques 14 (1969) 169-173 [in Italian].
- [8] G. Fubini, Sulle metriche definite da una forma Hermitiana, Atti del Reale Istituto Veneto di Science, Lettere ed Arti 63 (= 8. Ser., 6), 2. Part (1903-1904) 501-513. Reprinted in: G. Fubini, Opere Scelte [Selected Works], Vol. 2. Edizioni Cremonese, Rome, 1958, pp. 10-20 [in Italian].
- [9] E. Study, Kürzeste Wege im komplexen Gebiet, *Mathematische Annalen* **60** (1905) 321-377 [in German].
- [10] J. L. Coolidge, Hermitian metrics, Annals of Mathematics, 2. Ser., 22 (1920-1921) 11-28.
- [11] J. L. Coolidge, *The Geometry of the Complex Domain*, Clarendon Press, Oxford, 1924.
- [12] W. Blaschke and H. Terheggen, Trigonometria hermitiana, Rendiconti del Seminario Matematico della R. Università di Roma, 4. Ser., 3 (1939) 153-161.
 Reprinted in: W. Blaschke, Gesammelte Werke [Collected Works], Vol. 1, Eds. W. Burau et al.. Thales-Verlag, Essen, 1982, pp. 277-285 [in Italian].
- [13] E. Kasner, Conformality in connection with functions of two complex variables, Transactions of the American Mathematical Society 48 (1940) 50-62.
- [14] E. Kasner and J. de Cicco, Pseudo-conformal geometry: functions of two complex variables, *Bulletin of the American Mathematical Society* **48** (1942) 317-328.
- [15] J. de Cicco, The pseudo-angle in space of 2n dimensions, Bulletin of the American Mathematical Society 51 (1945) 162-168.
- [16] J. de Cicco and J. Synowiec, Elements of linear polygenic transformations and pseudo-angles of a complex vector space, *Rivista di Matematica della Università di Parma*, 2. Ser., **10** (1969) 55-70.
- [17] G. Giraud, Sur certaines fonctions automorphes de deux variables, *Annales Scientifiques de l'École Normale Supérieure*, 3. Ser., **38** (1921) 43-164 [in French].
- [18] P. A. Shirokov, *Tenzornoe Ischislenie*. Algebra Tenzorov [Tensor Calculus. Tensor Algebra], 2. ed.. Isdatel'stvo Kazanskogo Universiteta, Kazan, 1961 (first edition: 1934) [in Russian].

- [19] S. S. Chern and J. G. Wolfson, Minimal surfaces by moving frames, *American Journal of Mathematics* **105** (1983) 59-83.
- [20] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. II. Interscience Tracts in Pure and Applied Mathematics, No. 15, Vol. II. Interscience Publishers, New York, 1969.
- [21] K. Yano and I. Mogi, On real representations of Kaehlerian manifolds, *Annals of Mathematics*, 2. Ser., **61** (1955) 170-189.
- [22] P. A. Shirokov, Ob odnom tipe simmetricheskikh prostranstv, *Matematicheskii Sbornik*, N. S., **41(83)** (1957) 361-372. Reprinted in: P. A. Shirokov, *Izbrannye Raboty po Geometrii [Selected Works in Geometry]*, Isdatel'stvo Kazanskogo Universiteta, Kazan, 1966, pp. 408-418 [in Russian].
- [23] B. A. Rozenfel'd, K teorii simmetricheskikh prostranstv ranga 1, *Matematicheskii Sbornik*, N. S., **41(83)** (1957) 373-380 [in Russian].
- [24] B. A. Rozenfel'd, L. A. Vlasova, and T. I. Yukhtina, Primenenie ugla naklona 2-ploshchadki v ērmitovykh prostranstvakh k differentsial'noĭ geometrii ētikh prostranstv, *Izvestiya Vysshikh Uchebnykh Zavedeniĭ*, *Matematika* (28:)7(=266) (1984) 64-70. English translation: An application of the angle of inclination of a 2-area in Hermitian spaces to the differential geometry of these spaces, *Soviet Mathematics* (*Iz. VUZ*) 28:7 (1984) 81-91.
- [25] B. A. Rozenfel'd, Ērmitova trigonometriya P. A. Shirokova i ee obobshcheniya, *Izvestiya Vysshikh Uchebnykh Zavedeniĭ*, *Matematika* (**39**:)5(=396) (1995) 3-10. English translation: P. A. Shirokov's Hermitian trigonometry and its generalizations, *Russian Mathematics* (*Iz. VUZ*) **39**:5 (1995) 1-7.
- [26] G. B. Rizza, Deviazione caratteristica delle faccette piane di una varietà a struttura complessa, Atti della Accademia Nazionale dei Lincei, Rendiconti, Classe di Scienze Fisiche, Matematiche e Naturali, 8. Ser., 24 (1958 (= Anno 355)) 662-671 [in Italian].
- [27] G. B. Rizza, Holomorphic deviation for the 2q-dimensional sections of complex analytic manifolds, *Archiv der Mathematik* **10** (1959) 170-173.
- [28] B. Y. Chen, Slant immersions, Bulletin of the Australian Mathematical Society 41 (1990) 135-147.
- [29] B. Y. Chen and Y. Tazawa, Slant surfaces of codimension two, Annales de la Faculté des Sciences de Toulouse, Mathématiques, 5. Ser., 11:3 (1990) 29-43.

- [30] B. Y. Chen, Geometry of Slant Submanifolds, Katholieke Universiteit Leuven, Leuven, 1990.
- [31] S. Tsuchiya and M. Kobayashi, Some conditions for constancy of the holomorphic sectional curvature, $K\bar{o}dai\ Mathematical\ Seminar\ Reports\ {\bf 27}\ (1976)$ 379-384.
- [32] K. Scharnhorst, A special irreducible matrix representation of the real Clifford algebra C(3,1), Journal of Mathematical Physics 40 (1999) 3616-3631.
- [33] S. Kwietniewski, Ueber Flächen des vierdimensionalen Raumes, deren sämtliche Tangentialebenen untereinander gleichwinklig sind, und ihre Beziehung zu den ebenen Kurven [On surfaces of four-dimensional space, whose all tangential planes are isoclinic to each other, and their relation to the plane curves]. Dissertation. Verlag von E. Speidel, Zürich, 1902 [in German].
- [34] T. Maruyama, Some properties on geometry in complex space (part I), Journal of Science of the Gakugei Faculty, Tokushima University 1 (1950) 31-35 (Note, that a related, but somewhat different paper with a very similar title and by the same author exists: On the geometry in a complex space (part I), Scientific Papers of the Faculty of Engineering, Tokushima University 3:1 (1951) 1-3 [in Japanese, English abstract]).
- [35] Y. C. Wong, *Linear Geometry in Euclidean 4-Space*. Southeast Asian Mathematical Society Monograph No. 1. Southeast Asian Mathematical Society, Hong Kong, 1977.
- [36] M. Bruni, Relazioni tra metrica euclidea ed hermitiana in uno spazio vettoriale quaternionale, Atti della Accademia Nazionale dei Lincei, Rendiconti, Classe di Scienze Fisiche, Matematiche e Naturali, 8. Ser., 38 (1965 (= Anno 362)) 488-491 [in Italian, English abstract].
- [37] M. Bruni, Sulla nozione di deviazione caratteristica nello spazio vettoriale quaternionale, Atti della Accademia Nazionale dei Lincei, Rendiconti, Classe di Scienze Fisiche, Matematiche e Naturali, 8. Ser., 39 (1965 (= Anno 362)) 431-437 [in Italian, English abstract].
- [38] M. Bruni, Su alcune proprietà di geometria euclidea ed hermitiana in uno spazio vettoriale quaternionale, *Annali di Matematica Pura ed Applicata*, 4. Ser., **72** (1966) 60-77 [in Italian].
- [39] M. Bruni, Forme lineari generalizzate e forma di Kähler in una varietà a struttura quaternionale generalizzata, *Rivista di Matematica della Università di Parma*, 2. Ser., **7** (1966) 177-184 [in Italian, English abstract].

- [40] B. A. Rozenfel'd and R. P. Vyplavina, Ugly i orbity naklonov veshchestvennykh 2-ploshchadok v ērmitovykh prostranstvakh nad tenzornymi proizvedeniyami tel, *Izvestiya Vysshikh Uchebnykh Zavedenii*, *Matematika* (28:)7(=266) (1984) 70-74. English translation: Angles and unit vectors of inclination for real 2-areas in Hermitian spaces over a tensor product of fields, *Soviet Mathematics* (*Iz. VUZ*) 28:7 (1984) 92-96.
- [41] T. I. Yukhtina, Ugly i orty naklona 2-ploshchadok v ērmitovykh antikvaternionykh prostranstvakh, *Trudy Geometricheskogo Seminara (Kazan)* **22** (1994) 134-140 [in Russian].
- [42] G. B. Rizza, Deviazione caratteristica e proprietà locali delle 2q-faccette di una V_{2n} a struttura complessa, *Rendiconti, Accademia Nazionale dei XL*, 4. Ser., **10** (1959) 191-205 [in Italian].
- [43] E. Martinelli, Metrica hermitiana e metriche euclidea e simplettica associate, Rendiconti di Matematica, 6. Ser., 2 (1969) 295-313 [in Italian].
- [44] M. Bruni, Potenze esterne di metriche hermitiane in uno spazio vettoriale quaternionale, Atti della Accademia Nazionale dei Lincei, Rendiconti, Classe di Scienze Fisiche, Matematiche e Naturali, 8. Ser., 48 (1970 (= Anno 367)) 43-49 [in Italian].
- [45] M. Bruni, Misure euclidee, hermitiane, simplettiche e potenze esterne di uno spazio vettoriale quaternionale, *Annali di Matematica Pura ed Applicata*, 4. Ser., **88** (1971) 71-97 [in Italian].
- [46] M. Bordoni, Prodotti hermitiani nelle potenze esterne di uno spazio vettoriale reale a struttura complessa, Rendiconti di Matematica e delle sue Applicazioni,
 7. Ser., 4 (1984) 647-658 [in Italian].
- [47] S. Ianus and G. B. Rizza, Submanifolds of constant holomorphic deviation, *Bolletino della Unione Matematica Italiana*, 7. Ser., **B 11**, Suppl. 2 (1997) 115-124.
- [48] L. Bassotti and G. B. Rizza, Funzioni monogene nelle algebre reali di dimensione due e applicazioni conformi, *Rendiconti di Matematica*, 6. Ser., **6** (1973) 381-397 [in Italian].
- [49] G. B. Rizza, Monogenic functions on the real algebras and conformal mappings, Bolletino della Unione Matematica Italiana, 4. Ser., 12, Suppl. 3 (1975) 435-450.

- [50] E. V. Pavlov, Veshchestvennaya realizatsiya konformnogo sootvetstviya rimanovykh prostranstv nad kliffordovoĭ algebroĭ, *Izvestiya Vysshikh Uchebnykh Zavedeniĭ*, *Matematika* (22:)7(=194) (1978) 64-67. English translation: A real realization of conformal congruence of Riemannian spaces over a Clifford algebra, *Soviet Mathematics* (*Iz. VUZ*) 22:7 (1978) 52-55.
- [51] A. Herzer, Der äquiforme Raum einer Algebra, Mitteilungen der Mathematischen Gesellschaft in Hamburg 13 (1993) 129-154 [in German].